On the Dilation of Delaunay Triangulations of Points in Convex Position

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Abstract

Let S be a finite set of points in the Euclidean plane, and let $\mathcal{E}$ be the complete graph whose point-set is S. Chew, in 1986, proved a lower bound of $\pi/2$ on the stretch factor of the Delaunay triangulation of S (with respect to $\mathcal{E}$), and conjectured that this bound is tight. Dobkin, Friedman, and Supowit, in 1987, showed that the stretch factor of the Delaunay triangulation of S is at most $\pi(\sqrt{5}+1)/2 \approx 5.084$. This upper bound was later improved by Keil and Gutwin in 1989 to $2\pi/(3\cos(\pi/6)) \approx 2.42$. Since then (1989), Keil and Gutwin's bound has stood as the best upper bound on the stretch factor of Delaunay triangulations, even though Chew's conjecture is now widely believed to be true. Whether the stretch factor of Delaunay triangulations is $\pi/2$ or not remains a challenging and intriguing problem in computational geometry.

Bose, in an open-problem session at CCCG 2007, suggested looking at the special case when the points in S are in convex position.

In this paper we show that the stretch factor of the Delaunay triangulation of a point-set in convex position is at most $\rho = 2.33$.

1 Introduction

Let S be a finite set of points in the Euclidean plane, and let $\mathcal{E}$ be the complete graph whose point-set is S. A *Delaunay triangulation* of S is a triangulation in which the circumscribed circle of every triangle contains no point of S in its interior [8]. It is well known that if the points in S are *in general position* (i.e., no four points in S are cocircular) then the Delaunay triangulation of S is unique [8]. To simplify the discussion, we shall assume that the Delaunay triangulation is unique, even though the results in this paper are not contingent on this assumption.

The *Delaunay graph* of S is defined as the plane graph whose point-set is S, and whose edges are the edges of the Delaunay triangulation of S. An alternative equivalent definition is:

Definition 1 ([8]) An edge XY is in the Delaunay graph of S if and only if there exists a circle through points X and Y whose interior is devoid of points of S.

A subgraph G of $\mathcal{E}$ is said to have *stretch factor* or *dilation* ρ with respect to $\mathcal{E}$, if for every two points P and Q in S, the shortest path from P to Q in G has length at most $\rho \cdot |PQ|$, where $|PQ|$ is the Euclidean distance between P and Q.

Chew [6] showed a lower bound of $\pi/2$ on the stretch factor of the Delaunay graph.[1] Dobkin, Friedman, and Supowit [9, 10] in 1987 showed that the Delaunay graph of a point-set S has stretch factor $(1+\sqrt{5})\pi/2 \approx 5.08$ with respect to $\mathcal{E}$. This ratio was improved by Keil and Gutwin [12, 13] in 1989 to $C_{del} = 2\pi/(3\cos(\pi/6)) \approx 2.42$, which currently stands as the best upper bound on the stretch factor of the Delaunay graph. Many researchers, however, believe, that the lower bound of $\pi/2$ established in [6] is also an upper bound on the stretch factor of the Delaunay graph (for example, see page 470 in [17]). Whether this belief is true or not remains one of the most challenging and intriguing open problems in computational geometry.

In addition to its theoretical interest, improving the current upper bound on the stretch factor of Delaunay graphs has a huge and direct impact on the problem of constructing geometric *spanners* of Euclidean graphs, which has significant applications in the area of wireless computing. A *spanner* of $\mathcal{E}$ is a spanning subgraph of $\mathcal{E}$ that has a constant stretch factor. The problem of constructing geometric spanners has been extensively studied within computational geometry, and much of the early work on spanners was done from that perspective (for example, see [1, 4, 7, 11, 13, 15, 17, 19]). More recently, wireless network researchers have approached the problem as well. Emerging wireless distributed system technologies, such as wireless ad-hoc and sensor networks, are often modeled as geometric graphs. Spanners are fundamental to wireless *distributed* systems because they represent topologies that can be used for efficient unicasting, multicasting, *and/or* broadcasting (see [4, 5, 11, 14, 16, 18], to name a few). For these applications, spanners are typically required to be planar because planarity is useful for efficient routing [4, 5, 11, 14, 18]. Therefore, the Delaunay graph, or spanning subgraphs of the Delaunay graph,

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[1] In the same paper [6], Chew proved that the stretch factor of the Delaunay graph under the L_1 norm is at most $\sqrt{10}$.

are ideal for such applications. As a matter of fact, many spanner constructions in the literature rely on extracting subgraphs of the Delaunay graph (see for example [4, 11, 15, 16, 18]). The stretch factors of these constructed spanners are a constant times[2] the stretch factor of the Delaunay graph C_{del} [4, 11, 15, 16, 18]. Therefore, improving the upper bound on the stretch factor of Delaunay graphs (C_{del}) will automatically improve the stretch factors of all such spanners.

In an open-problem session of the 19th Canadian Conference on Computational Geometry (CCCG 2007) [3], Bose suggested looking at the special case when the points in S are in convex position. While settling this particular case may not lead to improving the upper bound for the general case, it may, however, shed some light on the intrinsic difficulty of this problem. Moreover, convexity is a very natural assumption in computational geometry that is interesting per se.

In this paper we progress towards this goal by showing that the stretch factor of the Delaunay graph of a point-set in convex position is at most $\rho = 2.33$. The precise value of ρ is the root of the equation $\rho^3 - \rho - (\pi + \arctan((1-\rho^2)/\rho))\sqrt{\rho^4 - \rho^2 + 1} = 0$, in the interval $[1, \infty)$.

2 The Theorem

Let $\rho = 2.33$. In this section, we will prove the following theorem:

Theorem 2 *Let S be a finite set of points in convex position in the plane. The stretch factor of the Delaunay triangulation of S is at most ρ.*

Proof. Let D be the Delaunay triangulation of S. Let $P, Q \in S$ be two arbitrary points, and denote by $|PQ|$ the Euclidean distance between P and Q. Denote by $p(P,Q)$ a shortest path between P and Q in D, and by $|p(P,Q)|$ the weight of $p(P,Q)$, that is, the sum of the Euclidean distances between every two consecutive points on $p(P,Q)$. We shall prove that $|p(P,Q)| \leq \rho|PQ|$.

We proceed by induction on the rank of $|PQ|$ among all pairs of points in S. (We assume that ties are broken arbitrarily.) If the distance between P and Q is the smallest among all pairs of points in S, then the circle with diameter PQ contains no points of S in its interior. By Definition 1, PQ is an edge in D, and hence $|p(P,Q)| = |PQ| \leq \rho|PQ|$.

Now suppose that the statement is true for any pair of points whose distance is less than $|PQ|$. Assume that PQ is not an edge in D (otherwise we are done by the same token as above).

[2]This constant is usually the stretch factor of the spanning subgraph of the Delaunay graph with respect to the Delaunay graph itself.

Since the point-set S is convex, there exist two parallel lines L_P and L_Q, passing through P and Q, respectively, such that on one side of the horizontal line PQ—either above or below it, all points of S lie between L_P and L_Q. Without loss of generality, assume that all points of S *above* the line PQ lie between L_P and L_Q. Note that if this set of points is empty, then P and Q are connected in D by a horizontal path (possibly a single edge) of weight $|PQ|$, and the statement follows.

Let $T \in S$ be a point above PQ that maximizes the angle $\gamma = \angle PTQ$. By Lemma 1 of [13], there exists a path below PQ whose length is at most $(\gamma/\sin\gamma)|PQ|$. If $\gamma \leq 2.058$, then $\gamma/\sin\gamma \leq \rho = 2.33$, and we are done. Therefore, we can assume that $\gamma > 2.058 > \pi/2$ in the rest of the proof. Consequently, $|PT| < |PQ|$ and $|TQ| < |PQ|$, and by the inductive hypothesis, we have $|p(P,T)| \leq \rho|PT|$ and $|p(T,Q)| \leq \rho|TQ|$.

Lemma 3 *If $|p(P,T)| = |PT|$ or $|p(T,Q)| = |TQ|$, then $|p(P,Q)| \leq \rho|PQ|$.*

Proof. Suppose that $|p(P,T)| = |PT|$, and let $\theta = \angle TPQ$. Note that $\frac{|PT|}{|PQ|} = \frac{\sin(\pi-\gamma-\theta)}{\sin\gamma} = \frac{\sin(\gamma+\theta)}{\sin\gamma}$, and $\frac{|TQ|}{|PQ|} = \frac{\sin\theta}{\sin\gamma}$. We have

$$\begin{aligned} \frac{|p(P,Q)|}{|PQ|} &\leq \frac{|p(P,T)| + |p(T,Q)|}{|PQ|} \\ &\leq \frac{|PT| + \rho|TQ|}{|PQ|} \\ &= \frac{\sin(\gamma+\theta) + \rho\sin\theta}{\sin\gamma} \\ &= \cos\theta + \frac{\sin\theta(\cos\gamma + \rho)}{\sin\gamma}. \end{aligned} \tag{1}$$

Define the function $g(\gamma)$ of γ in the interval $(2.058, \pi)$ as follows: $g(\gamma) = (\cos\gamma + \rho)/\sin\gamma$. Since $\rho = 2.33$ and $2.058 < \gamma < \pi$, it is easy to verify that $g'(\gamma) = (-1 - \rho\cos\gamma)/\sin^2\gamma > 0$, and hence $g(\gamma)$ is an increasing function in the chosen interval. Since $\gamma < \pi - \theta$, we have $g(\gamma) < g(\pi - \theta)$. Therefore, $g(\gamma) = (\cos\gamma + \rho)/\sin\gamma < (\cos(\pi-\theta) + \rho)/\sin(\pi-\theta) = (-\cos\theta + \rho)/\sin\theta$. The last inequality, together with Inequality (1), gives $|p(P,Q)|/|PQ| \leq \cos\theta + g(\gamma)\sin\theta \leq \rho$.

The proof is analogous when $|p(T,Q)| = |TQ|$. □

By the above lemma, we may assume that T does not lie on L_P or on L_Q because otherwise, $|p(P,T)| = |PT|$ or $|p(T,Q)| = |TQ|$ and we are done.

Since the point-set S is convex, there exists a line L_T passing through T such that all other points in S are below L_T. If the line L_T passes through P or Q, then either $|p(P,T)| = |PT|$ or $|p(T,Q)| = |TQ|$, and by Lemma 3 we are done. So we can assume that L_T does not pass through P or Q in the rest of the proof.

Let M and N be the intersections of L_T with L_P and L_Q, respectively. By the above discussion, M and N

are above the line PQ and the non-degenerate triangles $\triangle PMT$, $\triangle TPQ$, and $\triangle TQN$ do not overlap. Let $\alpha = \angle MPT$, and $\beta = \angle PMT$. See Figure 1 for an illustration. It is easy to see that $0 < \alpha < \gamma$, $0 < \beta < \pi - \alpha$, and $0 < \theta < \pi - \gamma$. (Also note that $2.058 < \gamma < \pi$.)

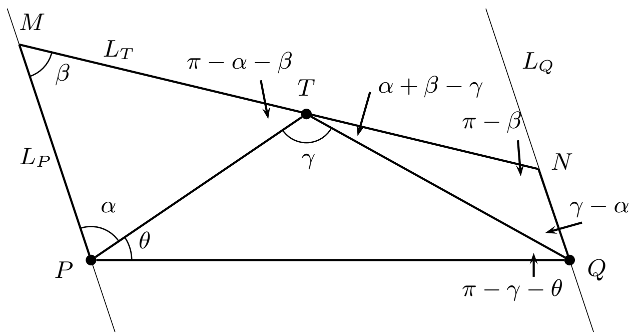

Figure 1: An illustration of the structure above PQ.

Since $TP \notin D$, and by convexity of S, there exists a path from P to T that is convex-away from PT and lies in the triangle $\triangle PMT$. By convexity, the length of this path is at most $|PM| + |MT|$ (see [2, p. 42]). Therefore, the length of the shortest path from P to T is at most $|PM| + |MT|$. Similarly, the length of the shortest path from T to Q is at most $|TN| + |NQ|$. Let $\rho_1 = \frac{|PM|+|MT|}{|PT|}$ and $\rho_2 = \frac{|TN|+|NQ|}{|TQ|}$. Then $|p(P,T)| \leq \rho_1 |PT|$ and $|p(T,Q)| \leq \rho_2 |TQ|$.

Lemma 4 $\rho_2 = \sin(\gamma - \alpha)\left(\frac{\sin\alpha}{\rho_1 - \cos\alpha}\right) + \cos(\gamma - \alpha)$.

Proof. We have:

$$\begin{aligned}
\rho_1 &= \frac{|PM| + |MT|}{|PT|} \\
&= \frac{\sin\alpha + \sin(\pi - \alpha - \beta)}{\sin\beta} \\
&= \frac{\sin\alpha + \sin\alpha\cos\beta + \cos\alpha\sin\beta}{\sin\beta} \\
&= \sin\alpha\left(\frac{1+\cos\beta}{\sin\beta}\right) + \cos\alpha. \qquad (2)
\end{aligned}$$

Equality (2) implies that $\frac{1+\cos\beta}{\sin\beta} = \frac{\rho_1 - \cos\alpha}{\sin\alpha}$, and hence

$$\frac{1-\cos\beta}{\sin\beta} = \frac{\sin\beta}{1+\cos\beta} = \frac{\sin\alpha}{\rho_1 - \cos\alpha}. \qquad (3)$$

On the other hand:

$$\begin{aligned}
\rho_2 &= \frac{|TN| + |NQ|}{|TQ|} \\
&= \frac{\sin(\gamma - \alpha) + \sin(\alpha + \beta - \gamma)}{\sin(\pi - \beta)} \\
&= \frac{\sin(\gamma - \alpha) + \sin(\alpha - \gamma)\cos\beta + \cos(\alpha - \gamma)\sin\beta}{\sin\beta} \\
&= \sin(\gamma - \alpha)\left(\frac{1-\cos\beta}{\sin\beta}\right) + \cos(\gamma - \alpha). \qquad (4)
\end{aligned}$$

Now plugging (3) into (4) we get $\rho_2 = \sin(\gamma - \alpha)\left(\frac{\sin\alpha}{\rho_1 - \cos\alpha}\right) + \cos(\gamma - \alpha)$. $\square$

Lemma 5 $\frac{|p(P,Q)|}{|PQ|} \leq \frac{\min(\rho_1,\rho)\sin(\gamma+\theta) + \min(\rho_2,\rho)\sin\theta}{\sin\gamma}$.

Proof. We have:

$$\begin{aligned}
\frac{|p(P,Q)|}{|PQ|} &\leq \frac{|p(P,T)| + |p(T,Q)|}{|PQ|} \\
&\leq \frac{\min(\rho_1,\rho)|PT| + \min(\rho_2,\rho)|TQ|}{|PQ|}.
\end{aligned}$$

Since $\frac{|PT|}{|PQ|} = \frac{\sin(\gamma+\theta)}{\sin\gamma}$ and $\frac{|TQ|}{|PQ|} = \frac{\sin\theta}{\sin\gamma}$, we have

$$\frac{|p(P,Q)|}{|PQ|} \leq \frac{\min(\rho_1,\rho)\sin(\gamma+\theta) + \min(\rho_2,\rho)\sin\theta}{\sin\gamma}.$$

$\square$

Lemma 6 *If* $\rho_1 \geq \rho$ *or* $\rho_2 \geq \rho$, *then* $\frac{|p(P,Q)|}{|PQ|} \leq \rho$.

Proof. Suppose that $\rho_1 \geq \rho$. By Lemma 4 we have:

$$\begin{aligned}
\rho_2 &= \sin(\gamma - \alpha)\left(\frac{\sin\alpha}{\rho_1 - \cos\alpha}\right) + \cos(\gamma - \alpha) \\
&\leq \sin(\gamma - a)\left(\frac{\sin\alpha}{\rho - \cos\alpha}\right) + \cos(\gamma - \alpha) \\
&= \frac{\sin(\gamma - \alpha)\sin\alpha + \rho\cos(\gamma - \alpha) - \cos(\gamma - \alpha)\cos\alpha}{\rho - \cos\alpha} \\
&= \frac{\rho\cos\gamma\cos\alpha + \rho\sin\gamma\sin\alpha - \cos\gamma}{\rho - \cos\alpha} \\
&= -\rho\cos\gamma + \frac{\cos\gamma(\rho^2 - 1) + \rho\sin\gamma\sin\alpha}{\rho - \cos\alpha} \\
&\leq -\rho\cos\gamma + \frac{\cos\gamma(\rho^2 - 1) + \rho\sin\gamma}{\rho - \cos\alpha}.
\end{aligned}$$

The above inequalities are true because $1 > \sin\alpha > 0$, $\sin\gamma > 0$, $\sin(\gamma - \alpha) > 0$, and $\rho - \cos\alpha > 0$.

Since $\rho = 2.33$ and $2.058 < \gamma < \pi$, it is easy to verify that $\cos\gamma(\rho^2 - 1) + \rho\sin\gamma$ is a decreasing function of γ. Therefore $\cos\gamma(\rho^2 - 1) + \rho\sin\gamma \leq (2.33^2 - 1)\cos(2.058) + 2.33\sin(2.058) \leq 0$. Also note that $\rho - \cos\alpha > 2.33 - 1 > 0$, and hence $\rho_2 \leq -\rho\cos\gamma + \frac{\cos\gamma(\rho^2-1)+\rho\sin\gamma}{\rho - \cos\alpha} \leq -\rho\cos\gamma$. Applying the last inequality to Lemma 5, we have:

$$\begin{aligned}
\frac{|p(P,Q)|}{|PQ|} &\leq \frac{\min(\rho_1,\rho)\sin(\gamma+\theta) + \min(\rho_2,\rho)\sin\theta}{\sin\gamma} \\
&\leq \frac{\rho\sin(\gamma+\theta) + \rho_2\sin\theta}{\sin\gamma} \\
&\leq \frac{\rho\sin(\gamma+\theta) - \rho\cos\gamma\sin\theta}{\sin\gamma} \\
&= \frac{\rho\sin\gamma\cos\theta + \rho\cos\gamma\sin\theta - \rho\cos\gamma\sin\theta}{\sin\gamma} \\
&= \rho\cos\theta \\
&\leq \rho. \qquad (5)
\end{aligned}$$

By symmetry, the same holds true if $\rho_2 \geq \rho$. This completes the proof of Lemma 6. $\square$

Lemma 7 *If* $1 \leq \rho_1, \rho_2 \leq \rho$, *then* $\frac{|p(P,Q)|}{|PQ|} \leq \rho$.

Proof. Since $\rho_1, \rho_2 \leq \rho$, by Lemma 5 we have:

$$\begin{aligned} \frac{|p(P,Q)|}{|PQ|} &\leq \frac{\min(\rho_1,\rho)\sin(\gamma+\theta) + \min(\rho_2,\rho)\sin\theta}{\sin\gamma} \\ &= \frac{\rho_1 \sin(\gamma+\theta) + \rho_2 \sin\theta}{\sin\gamma}. \end{aligned}$$

By Lemma 4, we have:

$$\frac{|p(P,Q)|}{|PQ|} \leq \frac{\rho_1 \sin(\gamma+\theta) + \frac{\sin(\gamma-\alpha)\sin\alpha\sin\theta}{\rho_1 - \cos\alpha} + \cos(\gamma-\alpha)\sin\theta}{\sin\gamma}.$$

For any fixed values of α, γ, θ, define the function $h(\rho_1)$ of ρ_1 in the interval $[1, \rho]$ as follows:

$$h(\rho_1) = \rho_1 \sin(\gamma+\theta) + \frac{\sin(\gamma-\alpha)\sin\alpha\sin\theta}{\rho_1 - \cos\alpha}.$$

Let $C_1 = \sin(\gamma+\theta)$, $C_2 = \frac{\sin(\gamma-\alpha)\sin\alpha\sin\theta}{\sin(\gamma+\theta)}$, and $C_3 = \sin(\gamma+\theta)\cos\alpha$. Then

$$h(\rho_1) = C_1(\rho_1 - \cos\alpha + \frac{C_2}{\rho_1 - \cos\alpha}) + C_3,$$

and its derivative is $h'(\rho_1) = C_1\left(1 - \frac{C_2}{(\rho_1 - \cos\alpha)^2}\right)$. Since $C_1, C_2, C_3 > 0$ are fixed and $\rho_1 - \cos\alpha > 0$, $h'(\rho_1)$ is a monotonically increasing function in the interval $[1, \rho]$. Thus by the *first derivative test* in calculus, the maximum value of the function $h(\rho_1)$ (and hence $\frac{|p(P,Q)|}{|PQ|}$) in the interval $[1, \rho]$ occurs on the boundary. When $\rho_1 = 1$, we have $|p(P,T)| = |PT|$ and by Lemma 3 we have $\frac{|p(P,Q)|}{|PQ|} \leq \rho$; and when $\rho_1 = \rho$, by Lemma 6 we have $\frac{|p(P,Q)|}{|PQ|} \leq \rho$. This completes the proof of Lemma 7. $\square$

Since $\rho_1, \rho_2 \geq 1$, combining Lemma 6 and Lemma 7, we conclude the proof of Theorem 2. $\square$

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