Local Algorithms for Constructing Spanners: Improved Bounds

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Abstract

Let S be a set of n points in the plane, let $\mathcal{E}$ be the complete Euclidean graph whose point-set is S, and let G be the Delauany triangulation of S. We present a very simple *local* algorithm that constructs a subgraph of G of degree at most 11 that is a geometric spanner of G with stretch factor 2.86. This algorithm gives an $O(n \lg n)$ time centralized algorithm for constructing a subgraph of G that is a geometric spanner of $\mathcal{E}$ of degree at most 11 and stretch factor < 7.

The algorithm can be generalized to unit disk graphs to give a local algorithm for constructing a plane spanner of a unit disk graph of degree at most 11 and stretch factor < 7.

1 Introduction

Let S be a set of points in the plane, and let $\mathcal{E}$ be the complete Euclidean graph whose point-set is S. It is well known that the Delaunay triangulation G of S is a plane geometric (i.e., with respect to the Euclidean distance) spanner of $\mathcal{E}$ with stretch factor $C_{del} < 2.42$ [12].

In this paper we consider the problem of constructing a bounded-degree subgraph of G that is a spanner of G under the *local* model of computation. The motivation behind such requirements on the subgraph stems from applications in wireless ad-hoc and sensor networks. In such applications plane spanners are used as the underlying topologies for efficient unicasting, multicasting, and broadcasting (e.g., see [4, 5, 9, 13, 15, 16, 18]). The bounded degree requirement is important for minimizing interference among the wireless devices in the network. A suitable model of computation for such systems is the *local* model, in which the computation performed by each device only depends on the information available within its neighborhood. More formally, a *local* algorithm is a distributed algorithm that can be simulated to run in a constant number of synchronous communication rounds [17].

Under the centralized model of computation, the problem of constructing a bounded-degree subgraph of G that is a spanner has received significant interest. Bose et al. [3, 4] were the first to show how to extract a subgraph of G that is a spanner of $\mathcal{E}$ with degree at most 27 and stretch factor 10.02. Bose et al. [6] then improved the aforementioned result and showed how to construct a subgraph of G that is a spanner of $\mathcal{E}$ with degree at most 17 and stretch factor 23. This result was subsequently improved by Kanj and Perković [10] who presented an algorithm that constructs a subgraph of G with degree at most 14 and stretch factor 3.54 (w.r.t. $\mathcal{E}$).[1] Very recently (unpublished), Carmi and Chaitman [7] were able to improve Kanj and Perković's result further by presenting an algorithm that computes a subgraph of G with degree at most 7 and stretch factor $(1 + \sqrt{2})^2 \cdot C_{del} < 14.1$.

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[1] A journal version of the results in [10] appears in [11] (with G. Xia).

All the aforementioned algorithms run in $O(n)$ time when G is given as input, and in time $O(n \lg n)$ otherwise ($n = |S|$).[2]

Under the local model of computation, Kanj and Perković's result [10] gives a local algorithm that computes a subgraph of G of degree at most 14 and stretch factor 3.54. Carmi and Chaitman's result [7] can be extended to the distributed model to yield a distributed algorithm that computes a subgraph of G of degree at most 7 and stretch factor 14.1; however, their algorithm is inherently nonlocal.

In this paper we present a very simple local algorithm that constructs a subgraph of G of degree at most 11 and stretch factor 2.86 with respect to G, and hence stretch factor $2.86 \cdot C_{del} < 7$ with respect to $\mathcal{E}$. The algorithm can be implemented to run in 2 synchronous communication rounds (i.e., the locality is 2). To put the result of this paper in context, this result improves the local algorithm of Kanj and Perković in terms of the minimum degree bound achieved (11 versus 14). Moreover, the algorithm presented in paper is simpler than that in [10].

The local algorithm presented in this paper can be implemented to run in $O(n)$ time under the centralized model when G is given, and in $O(n \lg n)$ time otherwise.

We note that in wireless computing, the network is often (modeled as a *unit disk graph* (UDG) rather than a complete Euclidean graph. Many of the algorithms mentioned above (among others) can be modified to construct bounded-degree plane spanners of UDGs [3, 4, 10, 11] (see also [18]). The results in this paper can be generalized to give a local algorithm for constructing a bounded-degree plane spanner of a UDG with the same upper bounds described above on the degree and the stretch factor.

2 Preliminaries

Given a set of points S in the 2-dimensional Euclidean plane, the complete Euclidean graph $\mathcal{E}$ on S is defined to be the complete graph whose point-set is S. Each edge ab connecting points a and b is assumed to be embedded in the plane as the straight line segment ab; the *weight* of ab is the Euclidean distance $|ab|$.

Let H be a subgraph of $\mathcal{E}$. The weight of a simple path $a = m_0, m_1, ..., m_r = b$ in H is $\sum_{j=0}^{r-1} |m_j m_{j+1}|$. A subgraph H' of H is said to be a *geometric spanner* of H if there is a constant ρ such that, for every two points $a, b \in H$, the weight of a shortest path from a to b in H' is at most ρ times the weight of a shortest path from a to b in H. The constant ρ is called the *stretch factor* of H' (with respect to H). The following is a well known—and obvious—fact:

Fact 2.1. *A subgraph H' of graph H has stretch factor ρ with respect to H if and only if for every edge $xy \in H$: the weight of a shortest path in H' from x to y is at most $\rho \cdot |xy|$.*

For three non-collinear points x, y, z in the plane we denote by $\bigcirc xyz$ the circumscribed circle of $\triangle xyz$. A *Delaunay triangulation* of S is a triangulation of S such that the circumscribed circle of every triangle in this triangulation (i.e., every triangular face) contains no point of S in its interior [8]. It is well known that if the points in S are *in general position* (no four points in S are cocircular) then the Delaunay triangulation of S is unique [8]. In this paper—as in most papers in the literature—we shall assume that the points in S are in general position; otherwise, the input can be slightly perturbed so that this condition is satisfied. The *Delaunay graph* of S is defined as the plane graph whose point-set is S and whose edges are the edges of the Delaunay triangulation of S. An alternative equivalent definition, usually referred to as the *empty circle* property, that we end up using is:

Definition 2.2 (The empty circle property). ([8]) An edge xy is in the Delaunay graph of S if and only if there exists a circle through points x and y whose interior contains no point in S.

[2]Very recently, a spanner of $\mathcal{E}$ of degree at most 6 and stretch factor 6 was given in [2]. This spanner, however, is not a subgraph of G.

It is well known that the Delaunay graph of S is a spanner of $\mathcal{E}$ with stretch factor $C_{del} \leq 4\sqrt{3}\pi/9 < 2.42$ [12].

Given integer parameter $k > 6$, the *Yao subgraph* [19] of a plane graph H is constructed by performing the following *Yao step*: For each point p in H partition the space (arbitrarily) into k cones of equal measure/size whose apex is p, thus creating k closed cones of angle $2\pi/k$ each, and choose the shortest edge in H out of p (if any) in each cone. The Yao subgraph consists of edges in H chosen by *either* endpoint. Note that the degree of a point in the Yao subgraph of H may be unbounded.

Let G be the Delaunay graph of S. Let ca and cb be edges in G such that $\angle bca \leq \theta$, for some angle θ.[3] If the interior of $\triangle cab$ is devoid of points of G, then it can be easily shown using the empty circle property (see Definition 2.2), that the interior of $\bigcirc cab$ below chord ab contains no points of G (for example, see Proposition 3.3 in [11]). In this case Keil and Gutwin [12] showed the following:

Lemma 2.3 (Lemma 1 in [12]). *If the interior of $\bigcirc abc$ below chord ab is devoid of points of S, then there exists a path from a to b in G, in the region interior to $\bigcirc abc$ above chord ab, whose weight is at most the length of arc $\widehat{ab}$.*

Note that if $ab \in G$ then the path described in Lemma 2.3 is simply the edge ab.

Let ca and cb be edges in G such that $|ca| \leq |cb|$, and suppose that the interior of $\triangle cab$ contains no points of S. Let $\mathcal{P} : (a = m_0, m_1, \ldots, m_k = b)$ be the path referred to in Lemma 2.3. The path $\mathcal{P}$ was called the *canonical path* between a and b in [10, 11], and the following structural properties about $\mathcal{P}$ were proved:

Lemma 2.4 ([10, 11]). *Let ca and cb be edges in G such that $\angle bca \leq \theta$, and such that ca is the shortest edge in the angular sector $\angle bca$. The canonical path $\mathcal{P} : (a = m_0, m_1, \ldots, m_k = b)$ in G satisfies:*

(i) $|ca| + \sum_{i=0}^{k-1} |m_i m_{i+1}| \leq (1 + \theta/\cos(\frac{\theta}{2}))|cb|$.

(ii) There is an edge from c to m_i, for $i = 0, \ldots, k$. Hence, if $ab \notin G$ then there is no edge in G between any pair m_i and m_j lying in the closed region enclosed by ca, cb and the edges of $\mathcal{P}$, for any i and j satisfying $0 \leq i < j \leq k$.

(iii) $\angle m_{i-1}m_im_{i+1} > \pi - \angle m_{i-1}cm_{i+1} > \pi - \theta$, *for* $i = 1, \ldots, k-1$.

Two edges mx, my incident to a point m in a subgraph H of $\mathcal{E}$ are said to be *consecutive* if one of the angular sectors determined by the two segments mx and my in the plane contains no neighbors of m.

The statement of the following lemma is well known and can be easily verified by the reader:

Lemma 2.5. *The function $\alpha/\sin(\alpha)$ is an increasing function in the interval $(0, \pi/2]$.*

Lemma 2.6. *Let $\widehat{yz}$ denote the arc facing angle $\angle yxz$ in $\bigcirc xyz$, and suppose that $\angle yxz \leq \theta$, where $\theta \in (0, \pi/2]$. Then $|\widehat{yz}|/|yz| = \angle yxz/\sin(\angle yxz) \leq \theta/\sin\theta$.*

Proof. The equality $|\widehat{yz}|/|yz| = \angle yxz/\sin(\angle yxz)$ is true by simple geometric arguments. The inequality $\angle yxz/\sin(\angle yxz) \leq \theta/\sin\theta$ follows from Lemma 2.5. □

The *unit disk graph* (UDG) on point-set S is the subgraph of $\mathcal{E}$ consisting of all edges xy with $|xy| \leq 1$.

[3] All angles in this paper are measured in radians.

Algorithm **Spanner**

1. for every wide sequence of edges around p, p selects the three edges in the sequence;
2. p partitions the remaining space around it (the space left after the sectors determined by the wide sequences are removed) into cones of apex p, each of size $\pi/5$ (note that the boundary cones might be of smaller size);
3. p selects the shortest edge in every nonempty cone, breaking ties arbitrarily;
4. for every empty cone around p, let pr and ps be the two consecutive edges incident to p such that the empty cone is contained within the sector $\angle rps$; if pr (resp. ps) has been already selected, then p selects ps (resp. pr); otherwise, p selects the longer edge between pr and ps breaking ties arbitrarily;
5. p keeps an edge pq if and only if pq is selected by both p and q;

Figure 1: The algorithm Spanner.

3 The spanner

Let G be the Delaunay graph of S. The basic idea behind the local algorithm is that every point selects at most 11 of its incident edges in G, and edges that are selected by both endpoints are kept; this guarantees that the degree of the resulting subgraph of G is at most 11. To ensure that the resulting subgraph is a spanner of G, we first guarantee that whenever an edge $pq \in G$ is not kept in the subgraph: (1) an edge pr is kept such that $|pr| \leq |pq|$ and $\angle rpq \leq \pi/5$, and (2) all edges on the canonical path from r to q, except possibly the first and the last edges are kept in the subgraph. Second, we use an inductive proof to show that even when the first and last edges on a canonical path are not kept, a "short" path between the endpoints of each of these two edges exists in the subgraph, which then can be used to upper bound the length of a path from r to q in the subgraph. Ensuring property (1) above requires an idea that seems counterintuitive at the surface: a longer edge incident to a point is selected in favor of a shorter consecutive edge in certain cases (step 4 of the algorithm **Spanner**, given in Figure 1). This favoritism also (implicitly) allows the inductive proof to go through (induction is now applied to "shorter" edges).

We start by presenting the local algorithm that constructs the subgraph of G and prove that it has degree at most 11 in Subsection 3.1. Then we proceed to prove an upper bound on its stretch factor in Subsection 3.2. Everything is then put together in Subsection 3.3.

3.1 The algorithm

The algorithm is presented in a way that emphasizes its locality: each point in G selects its candidate edges independently based only on its coordinates and the coordinates of its neighbors, and only edges that are selected by both their endpoints are kept in the spanner.

A sequence of three consecutive edges incident to a point p is said to be *wide* if the sum of the two angles formed by the two pairs of consecutive edges in this sequence is at least $4\pi/5$.

Every point $p \in G$ executes the algorithm **Spanner** given in Figure 1.

Definition 3.1. Point p *selects* an edge pq when point p selects pq in steps 1-4 of the algorithm **Spanner**. Point p *keeps* an edge pq when both p and q select pq.

Let G' be the subgraph of G consisting of the edges that are kept after the points in G have applied the algorithm **Spanner**.

Lemma 3.2. *Point p selects every edge of a wide sequence of edges around it.*

Proof. The statement directly follows form step 1 of the algorithm **Spanner**. □

Theorem 3.3. *The subgraph G' of G has degree at most 11.*

Proof. Since an edge in G is in G' if and only if the edge is selected by both its endpoints in the algorithm **Spanner**, it suffices to show that every point $p \in G$ selects at most 11 incident edges. Assume first that no edge is selected by p in step 1. In this case p will partition the space around it into exactly 10 cones of apex p, each of size $\pi/5$. For every nonempty cone $\mathcal{C}$, p selects in step 2 exactly one edge in $\mathcal{C}$—namely a shortest edge in $\mathcal{C}$. For every empty cone $\mathcal{C}$, p selects at most one edge in step 3, which can be "charged to", or associated with, the empty cone $\mathcal{C}$. It follows that in this case p selects at most 10 incident edges.

Suppose now that p selects some edges in step 1. We use a combinatorial argument to show that the total number of edges selected by p in this case is at most 11. The following terminology will be useful in the rest of the proof. Two wide sequences of edges around p are said to *overlap* if the two sequences share two edges; the two sequences are said to be *adjacent* if they share exactly one (boundary) edge, and the two sequences are said to be *disjoint* if they do not share any edges. We distinguish the following cases.

If there is exactly one wide sequence around p, then p selects the three edges in this sequence by Lemma 3.2, and partitions the remaining space around it, which measures at most $2\pi - 4\pi/5 = 6\pi/5$, into at most 6 cones. Any edge selected by p other than the wide-sequence edges can be corresponded in a one-to-one fashion with one of these 6 cones. It follows that p selects at most 9 edges.

If there are exactly two wide sequences around p, then the worst case happens when the angle of each sequence measures exactly $4\pi/5$, which is the smallest angle needed to form a wide sequence, and the two sequences overlap at two edges that form an angle which is almost equal to $4\pi/5$, thus costing us the selection of an extra edge for nothing. In this case p selects the 4 edges forming these two overlapping sequences in step 1 by Lemma 3.2. Then p partitions the remaining space into 6 cones, and selects at most 6 more edges in steps 2 and 3 that can be corresponded/charged in a one-to-one fashion to these 6 cones. It follows that p selects at most 10 incident edges in total.

If there are exactly three wide sequences around p, then note in this case that at least two of these sequences must overlap because the angle formed by each sequence is at least $4\pi/5$. Note also that it is impossible for each pair of these 3 sequences to overlap, and hence, at least two of these three sequences must be nonoverlapping (i.e., either disjoint or adjacent). As in the previous case, it is not difficult to verify that the worst case happens when each sequence measures exactly $4\pi/5$ and two sequences overlap at two edges that form an angle that is almost equal to $4\pi/5$. In this case p selects the 4 edges of the two overlapping sequences by Lemma 3.2, and the three edges of the third sequence. The remaining space around p measures at most $2\pi - 8\pi/5 = 2\pi/5$, and is partitioned into at most two parts. In the worst case, the two parts are nonempty and one part measures more than $\pi/5$, thus forcing the selection of two edges from it, and the other part measures less than $\pi/5$. In this case two cones will be placed in the larger part and one cone in the smaller part, and p selects at most 3 edges in addition to the wide-sequence edges. It follows that p selects at most 10 edges.

If there are exactly 4 wide sequences around p, then by a similar token to the above, the worst case happens when each sequence measures exactly $4\pi/5$, two sequences overlap at two edges that form an angle which is almost equal to $4\pi/5$, and the other two sequences also overlap at two edges that form an angle which is almost equal to $4\pi/5$. In this case p ends up selecting the 8 edges of the 4 sequences by Lemma 3.2, and the remaining area around p is partitioned into at most two parts that together measure at most $2\pi/5$. In the worst case the two parts are nonempty and one of them measures more than $\pi/5$, thus forcing the selection of two edges from it, and the other part measures less than $\pi/5$. In this case two cones will be placed in the larger part and one cone in the smaller part, and p will select at most 3 edges in addition to the wide-sequence edges. It follows that p selects at most 11 edges in this case. This scenario is depicted in Figure 2; this is the only case in which point p ends up selecting 11 edges. This completes the proof of the proposition. □

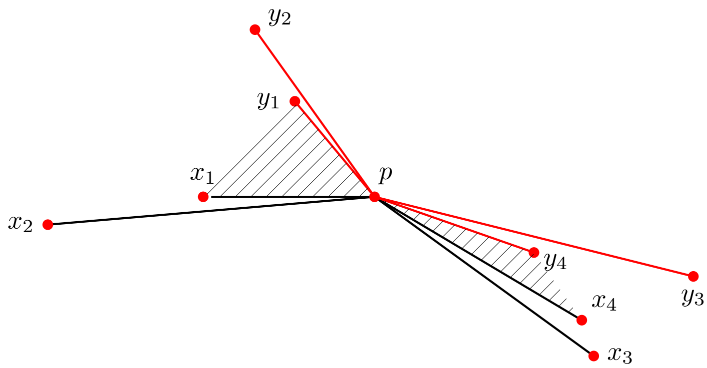

Figure 2: Illustration of the scenario in which point p selects 11 edges in the proof of Theorem 3.3. The two wide sequences px_1, px_2, px_3 and px_2, px_3, px_4 are overlapping, and p selects the 4 edges of the two sequences. Similarly, p selects the 4 edges of the two overlapping wide sequences py_1, py_2, py_3 and py_2, py_3, py_4. The angle $\angle x_1py_1$ measures more than $\pi/5$, and hence the shaded region between px_1 and py_1 will be partitioned into two cones, forcing p to select two edges, one from each cone, when these cones are nonempty. Finally, p selects one edge from the shaded region between px_4 and py_4 if this region is nonempty. Therefore, in this case p selects at most 11 edges.

3.2 The stretch factor

We start with the following lemmas:

Lemma 3.4. *Let xy and xz be two consecutive edges such that $|xy| > |xz|$ and $\angle yxz \geq 2\pi/5$. Then point x selects xy in the algorithm* **Spanner***.*

Proof. If xy is not selected by x in step 1 of the algorithm, then when x partitions the space around it into cones of apex x in step 2, at least one empty cone will be contained in the sector $\angle yxz$. This is true because each cone has size at most $\pi/5$ and $\angle yxz \geq 2\pi/5$. Since $|xy| > |xz|$, x is guaranteed to select xy in step 4 of the algorithm. □

Lemma 3.5. *Let xy and xz be two consecutive edges such that $\angle yxz \geq 3\pi/5$. Then point x selects both xy and xz in the algorithm* **Spanner***.*

Proof. If xy and xz are not edges of a wide sequence around x, then since $\angle yxz \geq 3\pi/5$, two empty cones defined in step 2 of the algorithm must fall within $\angle yxz$. When x considers these two empty cones in step 4, it will end up selecting both xy and xz. □

Lemma 3.6. *Let pq be an edge in G. If point p does not select pq in the algorithm* **Spanner***, then p selects an edge pr such that $|pr| \leq |pq|$ and $\angle rpq \leq \pi/5$. Moreover, edge pr is kept in G'.*

Proof. Suppose that p does not select pq. Since p does not select pq in step 1 of the algorithm, pq belongs to a cone $\mathcal{C}$ of apex p defined in step 2. Since p does not select pq in step 3, p must have selected an edge pr in $\mathcal{C}$ such that $|pr| \leq |pq|$. Since the size of $\mathcal{C}$ is at most $\pi/5$, $\angle rpq \leq \pi/5$.

To show that $pr \in G'$, since p selects pr, it suffices to show that pr is selected by point r in the algorithm **Spanner**. Let ps be the edge consecutive to pr in $\mathcal{C}$ (note that ps might be pq). Consider

$\triangle rps$, and note that since G is a triangulation and pr and ps are consecutive edges, all edges of $\triangle rps$ are edges in G. In particular, rp and rs are consecutive edges in G. If $\angle prs \geq 3\pi/5$, then by Lemma 3.5 applied to rp and rs, r selects rp and we are done. Assume now that $\angle prs < 3\pi/5$. Since $\angle rps \leq \angle rpq \leq \pi/5$, it follows that $\angle psr = \pi - \angle prs - \angle rpq > \pi/5$, and hence $|pr| > |rs|$. Since pr is a shortest edge in $\mathcal{C}$, $|pr| \leq |ps|$, which together with $\angle rps \leq \pi/5$, implies that $\angle prs \geq (\pi - \pi/5)/2 = 2\pi/5$. Now since $|rp| > |rs|$ and $\angle prs \geq 2\pi/5$, r selects rp by Lemma 3.4 applied to rp and rs. It follows that pr is kept in G'. □

For any two points p, q in S, denote by $d_{G'}(p,q)$ the weight of a shortest path between p and q in G'. To prove that the stretch factor of G', with respect to G, is at most $\rho = \frac{2\sin(2\pi/5)\cos(\pi/5)}{(2\sin(2\pi/5)\cos(\pi/5)-1)} < 2.86$, by Fact 2.1, it suffices to show that for every edge pq in G such that pq is not kept in G', $d_{G'}(p,q) \leq \rho|pq|$. (The choice of ρ will be justified in Proposition 3.12.) The proof is by induction on the rank of pq among all edges in G. The base case is when pq is the shortest edge in G. In this case if point p does not select edge pq in step 1 of the algorithm, edge pq will end up being the shortest edge in its cone, and hence will be selected in step 3 of the algorithm. Similarly, point q will also select edge pq, and hence pq is kept in G'. Therefore, $d_{G'}(p,q) = |pq| \leq \rho|pq|$. Now let pq be an edge in G, and assume by the inductive hypothesis that for every edge $xy \in G$ such that the rank of xy is strictly smaller than that of pq, there exists a path from x to y in G' of weight at most $\rho|xy|$. We will show that there exists a path from p to q in G' of weight at most $\rho|pq|$.

If pq is kept in G', then $d_{G'}(p,q) = |pq| \leq \rho|pq|$, and we are done. Therefore, we can assume in the rest of the proof that pq is not kept in G'. From step 5 in the algorithm **Spanner**, it follows that at least one of the points p, q does not select pq. Assume, without loss of generality, that p does not select pq. By Lemma 3.6, p selects an edge pr such that: $|pr| \leq |pq|$, $\angle rpq \leq \pi/5$, and pr is kept in G'. We will exhibit a path from r to q in G', which, together with edge pr, gives a path from p to q of weight at most $\rho|pq|$. We first consider the case when $rq \in G$. In this case we can induct on rq. We first need the following technical lemma whose proof is relegated to the appendix for lack of space:

Lemma 3.7 (Lemma 5.1, Appendix). *Let pr and pq be edges in G such that $\angle rpq \leq \pi/5$ and $|pr| \leq |pq|$. If $pr \in G'$ and $d_{G'}(r,q) \leq \rho|rq|$ then $d_{G'}(p,q) \leq \rho|pq|$.*

Proposition 3.8. *If rq is an edge in G then $d_{G'}(p,q) \leq \rho|pq|$.*

Proof. From $|pr| \leq |pq|$ and $\angle rpq \leq \pi/5$, it follows that $|rq| < |pq|$. Since $rq \in G$ and $|rq| < |pq|$, by the inductive hypothesis, we have $d_{G'}(r,q) \leq \rho|rq|$. Now $pr \in G'$, $\angle rpq \leq \pi/5$, and $d_{G'}(r,q) \leq \rho|rq|$, it follows from Lemma 3.7 that $d_{G'}(p,q) \leq \rho|pq|$. □

Suppose now that $rq \notin G$. We distinguish two cases based on whether or not the interior of $\triangle prq$ contains points of S. We need the following lemma:

Lemma 3.9 (Lemma 5.2, Appendix). *Let x, y, z be three points in S. Let $\alpha = \angle xyz$, $\beta = \angle yxz$, and $\gamma = \alpha + \beta$. If $\gamma \leq \pi/5$, $d_{G'}(y,z) \leq \frac{\pi}{5\sin(\pi/5)}|yz|$, and $d_{G'}(x,z) \leq \rho|xz|$, then $d_{G'}(x,y) \leq \rho|xy|$.*

Suppose that the interior of $\triangle prq$ contains no points of S. Consider the canonical path $\mathcal{P} : \langle m_0 = r, m_1, \ldots, m_k = q\rangle$ from r to q in G, defined in Section 2. Observe the following:

Observation 1. *Every internal point on $\mathcal{P}$ selects both edges incident to it on $\mathcal{P}$. Therefore, every edge on $\mathcal{P}$, except possibly the first and the last edges, are kept in G'.*

Proof. For any internal point m_i on $\mathcal{P}$ $(0 < i < k)$, $\angle m_{i-1}m_im_{i+1} \geq \pi - \pi/5 \geq 4\pi/5$ by part (iii) of Lemma 2.4. Therefore, both edges $m_{i-1}m_i$ and m_im_{i+1} are edges of a wide sequence of edges around m_i (note that $pm_i \in G$, for $i = 0, \ldots, k$, by part (*ii*) of Lemma 2.4). By Lemma 3.2, m_i selects both edges m_im_{i-1} and m_im_{i+1}.

Now for every edge on $\mathcal{P}$ other than the first and last edges, both its endpoints are internal points on $\mathcal{P}$. Therefore, both endpoints of this edge select the edge, and the edge is kept in G'. $\square$

By Observation 1, at most two edges on $\mathcal{P}$ are not kept in G'. We first consider the following two cases: exactly one edge of $\mathcal{P}$ is not kept in G', and exactly two edges of $\mathcal{P}$ are not kept in G'.

Lemma 3.10. *If exactly one edge of $\mathcal{P}$ is not kept in G' then $d_{G'}(r,q) \leq \rho|rq|$.*

Proof. If exactly one edge of $\mathcal{P}$ is not kept in G', then by Observation 1, this edge is either the first edge rm_1 or the last edge $m_{k-1}q$. Suppose that the edge that is not kept in G' is edge rm_1; the analysis of the other case is very similar. Let $\mathcal{P}'$ be the subpath of $\mathcal{P}$ from m_1 to m_k, that is, $\mathcal{P}$ minus edge rm_1, and note that all edges of $\mathcal{P}'$ are in G'. By Lemma 2.3, $wt(P')$ is at most the length of arc $\widehat{m_1q}$ facing angle $\angle m_1pq$ in $\bigcirc pm_1q$. Since $\angle m_1pq \leq \pi/5$, by Lemma 2.6, $|\ \widehat{m_1q}\ | \leq (\pi/(5\sin(\pi/5)))|m_1q|$. Therefore, $d_{G'}(m_1,q) \leq (\pi/(5\sin(\pi/5)))|m_1q|$. On the other hand, since m_1 is in the region of $\bigcirc prq$ subtended by chord rq and facing $\angle rpq$, we have $|rm_1| < |rq| < |pq|$. Since $rm_1 \in G$ and $|rm_1| < |pq|$, by the inductive hypothesis, we have $d_{G'}(r,m_1) \leq \rho|rm_1|$. Now consider $\triangle rm_1q$, and let $\alpha = \angle rqm_1$, $\beta = \angle qrm_1$, and $\gamma = \alpha + \beta$. Since m_1 is in the region of $\bigcirc prq$ subtended by chord rq, we have $\gamma = \alpha + \beta \leq \angle rpq \leq \pi/5$. Now consider the three points r, m_1, and q. Noting the previous facts, we can apply Lemma 3.9 with $x = r$, $y = q$, and $z = m_1$ to conclude that $d_{G'}(r,q) \leq \rho|rq|$. $\square$

Lemma 3.11. *If exactly two edges of $\mathcal{P}$ are not kept in G' then $d_{G'}(r,q) \leq \rho|rq|/(\sin(2\pi/5))$.*

Proof. If two edges of $\mathcal{P}$ are not kept in G', then by Observation 1 those edges are the first and last edges rm_1 and qm_{k-1}, respectively. We refer the reader to Figure 3 for illustration. Consider the subpath $\mathcal{P}''$ of $\mathcal{P}$ from m_1 to m_{k-1}, and note that all edges of $\mathcal{P}''$ are kept in G'. By Lemma 2.3, $wt(P'')$ is at most the length of arc $m_1\widehat{m}_{k-1}$ facing angle $\angle m_1pm_{k-1}$ in $\bigcirc pm_1m_{k-1}$ (note that $pm_1, pm_{k-1} \in G$ by part (ii) of Lemma 2.4). Since $\angle m_1pm_{k-1} \leq \pi/5$, by Lemma 2.6, $|\ m_1\widehat{m}_{k-1}\ | \leq (\pi/(5\sin(\pi/5)))|m_1m_{k-1}|$, and $wt(\mathcal{P}'') \leq (\pi/(5\sin(\pi/5)))|m_1m_{k-1}|$.

Since $|rm_1| < |pq|$, by the inductive hypothesis, $d_{G'}(r,m_1) \leq \rho|rm_1|$. We can now apply Lemma 3.9 to points r, m_1 and m_{k-1} to conclude that $d_{G'}(r,m_{k-1}) \leq \rho|rm_{k-1}|$ (it can be easily verified that the preconditions of the lemma are satisfied). On the other hand, since $|m_{k-1}q| < |pq|$, by the inductive hypothesis, we have $d_{G'}(m_{k-1},q) \leq \rho|m_{k-1}q|$. Now consider $\triangle rm_{k-1}q$ and observe that since pm_{k-1} is an edge in G (part (ii) in Lemma 2.4), we have $\angle rm_{k-1}q \geq \pi - \pi/5 = 4\pi/5$. Under the condition that $\angle rm_{k-1}q \geq 4\pi/5$ in $\triangle rm_{k-1}q$, it is not difficult to verify that $|rm_{k-1}| + |m_{k-1}q|$ is maximum when $\angle rm_{k-1}q = 4\pi/5$ and $|rm_{k-1}| = |m_{k-1}q|$. In this case we have $|rm_{k-1}| + |m_{k-1}q| \leq |rq|/(\sin(2\pi/5))$. It follows that in this case we have $d_{G'}(r,q) \leq d_{G'}(r,m_{k-1}) + d_{G'}(q,m_{k-1}) \leq \rho(|rm_{k-1}| + |m_{k-1}q|) \leq \rho|rq|/(\sin(2\pi/5))$. $\square$

Now we are ready to prove that $d_{G'}(p,q) \leq \rho|pq|$.

Proposition 3.12. *If the interior of $\triangle pqr$ contains no points of S then $d_{G'}(p,q) \leq \rho|pq|$.*

Proof. Consider the canonical path $\mathcal{P}$ from r to q. By Observation 1, at most two edges on $\mathcal{P}$ are not kept in G'. If all edges of $\mathcal{P}$ are kept in G', then since pr is kept in G', the path pr followed by $\mathcal{P}$ is a path from p to q in G' of weight $|pr| + wt(\mathcal{P}) \leq (1 + (\pi/5)\cos(\pi/10))|pq|$ by part (i) of Lemma 2.4, which is at most $\rho|pq|$ for the chosen value of ρ. Therefore, in this case we have $d_{G'}(p,q) \leq \rho|pq|$.

If exactly one edge of $\mathcal{P}$ is not kept in G', then by Lemma 3.10, we have $d_{G'}(r,q) \leq \rho|rq|$. Now $pr \in G'$, $|pr| \leq |pq|$, $\angle prq \leq \pi/5$, and $d_{G'}(r,q) \leq \rho|rq|$, it follows from Lemma 3.7 that $d_{G'}(p,q) \leq \rho|pq|$.

If exactly two edges of $\mathcal{P}$ are not kept in G', then by Observation 1 those edges must be rm_1 and qm_{k-1}. By Lemma 3.11, we have $d_{G'}(r,q) \leq \rho|rq|/(\sin(2\pi/5))$. Since rm_1 and qm_{k-1} were not kept in G', and since rm_1 was selected by m_1 and qm_{k-1} was selected by m_{k-1} (Observation 1), it follows

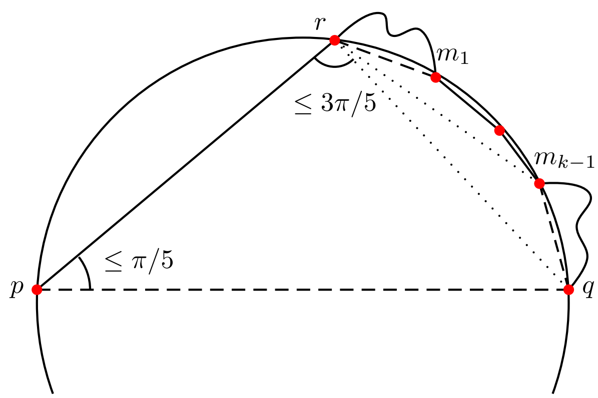

Figure 3: Illustration for the case when both edges rm_1 and $m_{k-1}q$ are not kept in G' in the proof of Lemma 3.11. Dashed lines indicate edges in G and solid lines indicate spanner edges.

that rm_1 was not selected by r and qm_{k-1} was not selected by q, in the algorithm **Spanner**. Since rp and rm_1 are consecutive edges at r, and pq and qm_{k-1} are consecutive edges at q (implied from part (*ii*) of Lemma 2.4 and the fact that G is a triangulation), by Lemma 3.5, it follows that each of $\angle prm_1$ and $\angle pqm_{k-1}$ is less than $3\pi/5$, which, in turn, implies that each of $\angle prq$ and $\angle pqr$ is less than $3\pi/5$. Consider $\triangle prq$. Since $|pr| \leq |pq|$, $\angle rpq \leq \pi/5$, and $\angle prq < 3\pi/5$, we conclude that $2\pi/5 \leq \angle prq \leq 3\pi/5$, and consequently, $\sin(\angle prq) \geq \sin(2\pi/5)$. Now $|rq| = (\sin(\angle rpq)/\sin(\angle prq))|pq| \leq (\sin(\pi/5)/\sin(2\pi/5))|pq| = |pq|/(2\cos(\pi/5))$. Since $d_{G'}(r,q) \leq \rho|rq|/(\sin(2\pi/5))$, it follows from the previous statement that $d_{G'}(r,q) \leq \rho|pq|/(2\sin(2\pi/5)\cos(\pi/5))$, and $d_{G'}(p,q) \leq |pr| + d_{G'}(r,q) \leq |pq| + d_{G'}(r,q) \leq (1 + \rho/(2\sin(2\pi/5)\cos(\pi/5))|pq| \leq \rho|pq|$. The last inequality is true if and only if $\rho \geq \frac{2\sin(2\pi/5)\cos(\pi/5)}{(2\sin(2\pi/5)\cos(\pi/5)-1)}$, which is satisfied by the chosen value of ρ. □

Proposition 3.13. *If the interior of $\triangle pqr$ contains points of S then $d_{G'}(p,q) \leq \rho|pq|$.*

Proof. Let S' be the set of points consisting of points r and q plus all points interior to $\triangle pqr$ (note that $p \notin S'$). Let $CH(S')$ be the set of points on the convex hull of S. Then $CH(S')$ consists of points $n_0 = r$ and $n_s = q$, and points $n_1, \ldots, n_{s-1}$ of G interior to $\triangle pqr$. Note that, by convexity, and because G is a triangulation, $pn_i \in G$, for $i = 0, \ldots, s$.

For $i = 0, \ldots, s-1$, the interior of $\triangle pn_in_{i+1}$ contains no points of G. Since $pn_i, pn_{i+1} \in G$, by Lemma 2.3 there exists a canonical path P_i from n_i to n_{i+1} in G. We argue next that at most one edge of P_i is not kept in G'.

Because n_i and n_{i+1} ($i = 0, \ldots, s-1$) are two consecutive points on $CH(S)$, and since the interior of $\triangle pqr$ is not empty, at least one of the two points n_i, n_{i+1} must be interior to $\triangle pqr$. Assume that n_i is interior to $\triangle pqr$; the proof is similar if n_{i+1} was interior (and n_i was not). If P_i consists of a single edge, then the statement that at most one edge of P_i is not kept in G' is vacuously true. Suppose now that P_i does not consist of a single edge, and consider the first point, x, after n_i on P_i. By Observation 1, at most two edges on the canonical path P_i, namely the first and the last edges (note that $\angle n_i p n_{i+1} \le \pi/5$), are possibly not kept in G'. Therefore, to show that at most one edge on P_i is not kept in G', it suffices to show that the first edge $n_i x$ on P_i is kept in G'. Since x is an internal point on P_i, by Observation 1, x selects edge xn_i. On the other hand, n_i is a point on $CH(S)$ that is interior to $\triangle pqr$. Therefore, the angle formed by the last edge yn_i on P_{i-1} and $n_i x$ is $> \pi$. Consequently, the edges $n_i y, n_i p, n_i x$ are edges of a wide sequence around n_i, and n_i selects $n_i x$ by Lemma 3.2. Therefore, $n_i x \in G'$ and at most one edge on P_i is not kept in G'.

Now $pn_i, pn_{i+1} \in G$, $\angle n_i p n_{i+1} \le \pi/5$, and at most one edge of P_i is not in G', by Lemma 3.10 applied to n_i and n_{i+1}, we obtain $d_{G'}(n_i, n_{i+1}) \le \rho |n_i n_{i+1}|$. It follows that $d_{G'}(r,q) = d_{G'}(n_0, n_s) \le \sum_{i=0}^{s-1} d_{G'}(n_i, n_{i+1}) \le \rho \sum_{i=0}^{s-1} |n_i n_{i+1}|$.

Extend rn_1 and qn_{s-1}; by convexity, rn_1 and qn_{s-1} meet at a point t inside $\triangle rpq$ (note that if $n_1 = n_s$, and hence there is exactly one point inside $\triangle pqr$, then $t = n_1 = n_{s-1}$). By convexity [1], we have $\sum_{i=0}^{s-1} |n_i n_{i+1}| \le |rt| + |tq|$. We will now upper bound $|rt| + |tq|$. Please refer to Figure 4 for illustration.

Since $|pr| \le |pn_1|$ and t is on the extension of rn_1, we have $|pt| \ge |pr|$. If t' is the intersection point of rt and pq, then by the triangular inequality we have $|rt| + |tq| \le |rt'| + |t'q|$. Therefore, we may assume that t is on pq. Moreover, since $|pt| \ge |pr|$, $|rt| + |tq|$ is largest when $|pr| = |pt|$ (this corresponds to $t = t''$ in Figure 4). In this case we have $|rt| + |tq| \le 2|pr|\sin(\pi/10) + |pq| - |pr|$ (since $|rt| \le 2|pr|\sin(\pi/10)$). Now $d_{G'}(p,q) \le |pr| + d_{G'}(r,q) \le |pr| + \rho \sum_{i=0}^{s-1} |n_i n_{i+1}| \le |pr| + \rho(|rt| + |tq|) \le |pr| + \rho(2|pr|\sin(\pi/10) + |pq| - |pr|) \le \rho|pq|$. The last inequality is true because $\rho \ge 1/(1 - 2\sin(\pi/10))$ (after simplification). □

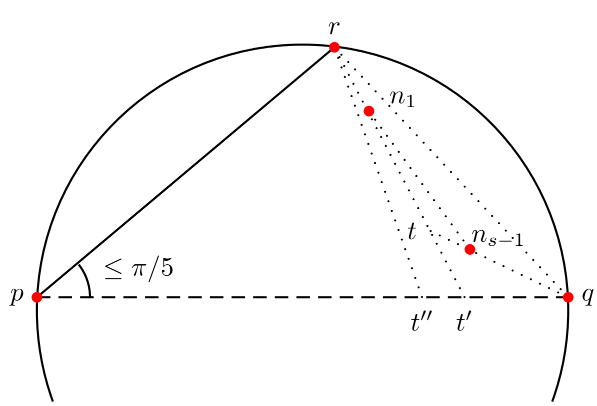

Figure 4: Illustration for the proof of Proposition 3.13.

Combining Proposition 3.8, Proposition 3.12, and Proposition 3.13, we conclude:

Theorem 3.14. *The subgraph G' is a spanner of G with stretch factor (w.r.t. G) $\rho = \frac{2\sin(2\pi/5)\cos(\pi/5)}{(2\sin(2\pi/5)\cos(\pi/5)-1)} < 2.86$.*

3.3 Putting it together

Combining the results in Subsection 3.1 and Subsection 3.2, we have:

Theorem 3.15. *The algorithm* **Spanner** *is a 2-local algorithm that, given the Delaunay triangulation G of a point-set S, computes a subgraph of G of degree at most 11 that is a spanner of G with stretch factor $\rho = \frac{2\sin(2\pi/5)\cos(\pi/5)}{2\sin(2\pi/5)\cos(\pi/5)-1} < 2.86$. The processing time for each $p \in S$ in the algorithm is linear in the degree of p.*

Proof. Let G' be the subgraph of G consisting of the set of edges that are selected by both endpoints after the application of the algorithm **Spanner**. By Theorem 3.3, the degree of G' is at most 11. By Theorem 3.14, G' is a spanner of G with stretch factor (w.r.t. G) $\rho = \frac{2\sin(2\pi/5)\cos(\pi/5)}{(2\sin(2\pi/5)\cos(\pi/5)-1)} < 2.86$.

Now to see that the algorithm is a 2-local algorithm, observe that the algorithm can be implemented in 2 synchronous communication rounds. In the first round, each point sends its coordinates to its neighbors. In the second round, each point p selects some edges incident on it according to steps 1-4 in the algorithm **Spanner**; then p informs each neighbor q whether it has selected edge pq or not. A point p keeps an edge pq if p has selected pq and it has received a message from its neighbor q (in the second round) indicating that q has selected pq as well. Finally, to see that the processing time at a point p is linear in the degree δ_p of p in G, observe first that p can determine the edges in its wide sequences as follows. Point p partitions the space around it into 10 cones of apex p, each of size $\pi/5$. Since the number of cones is constant, in linear time in δ_p, p can determine those cones that are empty. Since the total angle of a wide sequence of edges is at leat $4\pi/5$, for any wide sequence around p, an empty cone must fall within two consecutive edges in this sequence. Therefore, p can use the empty cones to determine the edges of the wide sequences around it. After determining the edges of the wide sequences, p partitions the reaming space around it into at most 10 cones, and for each cone, determines the set of its incident edges that fall in the cone. Then, the shortest edge in every cone can be found in time $O(\delta_p)$. Finally, since the number of empty cones around p is a constant, step 4 can be performed in time $O(\delta_p)$. The proof follows. ☐

Corollary 3.16. *Given a set S of n points in the plane, there exists an $O(n \lg n)$ time (centralized) algorithm that computes a spanner G' of the complete Euclidean graph $\mathcal{E}$ on point-set S, such that G' is a subgraph of the Delaunay triangulation of S, G' has degree at most 11, and G' has stretch factor $\rho \cdot C_{del} < 7$ with respect to $\mathcal{E}$, where C_{del} is the stretch factor of the Delaunay triangulation of S with respect to $\mathcal{E}$.*

Proof. The algorithm starts by computing the Delaunay triangulation G of S in time $O(n \lg n)$ (see [8]), and then feeds G to the algorithm **Spanner**. Noting that the stretch factor of G (with respect to $\mathcal{E}$) is $C_{del} < 2.42$, the statement of the theorem then follows from Theorem 3.15. ☐

4 Unit Disk Graphs

Unit disk graphs (UDGs) are very important in wireless computing since they have been widely used as a theoretical model for wireless networks. In particular, the problem of constructing bounded-degree plane spanners of UDGs has received a lot of interest [6, 11, 14, 15]. In such applications the local model of computation is a suitable working model because the wireless devices have limited energy, and lack the centralized coordination.

The results in Section 3 do not directly give a local algorithm for constructing a bounded-degree plane spanner of a UDG U for two reasons. The first reason is that the Delaunay triangulation of the point-set of U cannot be computed locally, and the second reason is that not every edge in the Delaunay triangulation of the point-set of U is an edge in U (i.e., has length at most 1 unit).

To overcome the above-mentioned obstacles, Wang et al. [14, 15] introduced a plane subgraph of U, called $LDel^{(2)}(U)$, that can be computed by a 3-local algorithm [11], and that contains all Delaunay edges of length at most 1, in addition to (possibly) some other edges. Moreover, they proved that the stretch factor of $LDel^{(2)}(U)$ is at most $C_{del} < 2.42$. Subsequently, $LDel^{(2)}(U)$ has been used as the underlying subgraph in several local and distributed algorithms that construct bounded-degree plane spanners of UDGs (see [11, 14, 15], to name a few).

We can use $LDel^{(2)}(U)$ as the underlying subgraph G in the algorithm **Spanner** to obtain the following result:

Theorem 4.1. *There exists a 3-local algorithm that, given a connected UDG U on n points, computes a plane subgraph of U of degree at most 11 that is a spanner of the complete Euclidean graph on the point-set of U, and that has stretch factor $\rho \cdot C_{del} = \frac{2\sin(2\pi/5)\cos(\pi/5)}{2\sin(2\pi/5)\cos(\pi/5)-1} \cdot C_{del} < 7$. The processing time for each $p \in U$ in the algorithm is $O(\delta_p \lg \delta_p)$, where δ_p is the degree of p in U.*

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5 Appendix

Lemma 5.1. *Let pr and pq be edges in G such that $\angle rpq \leq \pi/5$ and $|pr| \leq |pq|$. If $pr \in G'$ and $d_{G'}(r,q) \leq \rho|rq|$ then $d_{G'}(p,q) \leq \rho|pq|$.*

Proof. Let $\alpha = \angle qpr$ and $\beta = \angle rqp$. Since $pr \in G'$ and $d_{G'}(r,q) \leq \rho|rq|$, we have $d_{G'}(p,q) \leq |pr| + \rho|rq|$. Therefore, it suffices to show that $|pr| + \rho|rq| \leq \rho|pq|$. We have:

$$\begin{aligned} & |pr| + \rho|rq| \leq \rho|pq| \\ \Leftrightarrow\ & \sin\beta + \rho\sin\alpha \leq \rho\sin(\alpha+\beta) \\ \Leftrightarrow\ & \sin\beta \leq \rho(\sin(\alpha+\beta) - \sin\alpha) \\ \Leftrightarrow\ & \frac{\sin\beta}{\sin(\alpha+\beta)-\sin\alpha} \leq \rho. \end{aligned}$$

The last inequality is true because $\alpha \leq \pi/5$ and $|pq| \geq |pr|$, which together imply that $\beta \leq \pi/2 - \alpha/2$, and hence $\sin(\alpha+\beta) > \sin\alpha$. Using trigonometric identities we can derive that:

$$\frac{\sin\beta}{\sin(\alpha+\beta)-\sin\alpha} = \frac{1}{\cos\alpha - \tan(\beta/2)\sin\alpha}.$$

Since $\alpha \leq \pi/5$, $\beta/2 \leq \pi/4 - \alpha/4$, $\cos\alpha$ is decreasing in $[0,\pi/5]$, $\sin\alpha$ is increasing in $[0,\pi/5]$, and $\tan(\beta/2)$ is increasing in $[0,\pi/2)$, we have:

$$\begin{aligned} \frac{1}{\cos\alpha - \tan(\beta/2)\sin\alpha} &\leq \frac{1}{\cos(\alpha) - \tan(\pi/4 - \alpha/4)\sin\alpha} \\ &= \frac{\cos(\pi/4-\alpha/4)}{\cos(\pi/4+3\alpha/4)} \leq \cos(\pi/5)/\cos(2\pi/5) \leq \rho. \end{aligned}$$

The inequality before the last follows from the facts that $\alpha \leq \pi/5$ and the cosine function is decreasing in $[0,\pi/2]$. □

Lemma 5.2. *Let x, y, z be three points in S. Let $\alpha = \angle xyz$, $\beta = \angle yxz$, and $\gamma = \alpha + \beta$. If $\gamma \leq \pi/5$, $d_{G'}(y,z) \leq \frac{\pi}{5\sin(\pi/5)}|yz|$, and $d_{G'}(x,z) \leq \rho|xz|$, then $d_{G'}(x,y) \leq \rho|xy|$.*

Proof. Since $d_{G'}(x,y) \leq d_{G'}(x,z) + d_{G'}(z,y)$, from the statement of the lemma it follows that $d_{G'}(x,y) \leq \frac{\pi}{5\sin(\pi/5)}|yz| + \rho|xz|$. Therefore, it suffices to show that $\frac{\pi}{5\sin(\pi/5)}|yz| + \rho|xz| \leq \rho|xy|$. We have:

$$\begin{aligned} & \frac{\pi}{5\sin(\pi/5)}|yz| + \rho|xz| \leq \rho|xy| \\ \Leftrightarrow\ & \frac{\pi}{5\sin(\pi/5)}\sin\beta + \rho\sin\alpha \leq \rho\sin\gamma \quad \text{(using trigonometric relations in } \triangle xyz\text{)} \\ \Leftrightarrow\ & \frac{\pi}{5\sin(\pi/5)}\sin\beta \leq \rho(\sin\gamma - \sin\alpha) \\ \Leftrightarrow\ & \frac{\frac{\pi}{5\sin(\pi/5)}\sin\beta}{\sin\gamma - \sin\alpha} \leq \rho \quad (\text{because} \sin\gamma > \sin\alpha) \\ \Leftrightarrow\ & \frac{\frac{\pi}{5\sin(\pi/5)}}{\frac{\sin\gamma - \sin(\gamma-\beta)}{\sin\beta}} \leq \rho \\ \Leftrightarrow\ & \frac{\frac{\pi}{5\sin(\pi/5)}}{\sin\gamma(\frac{1-\cos\beta}{\sin\beta}) + \cos\gamma} \leq \rho \\ \Leftrightarrow\ & \frac{\frac{\pi}{5\sin(\pi/5)}}{\sin\gamma\tan\frac{\beta}{2} + \cos\gamma} \leq \rho. \end{aligned}$$

Since $\sin\gamma\tan\frac{\beta}{2} \geq 0$ (both $\beta, \gamma \in [0, \pi/5]$), and since the function $\cos x$ is a decreasing function in $(0, \pi/2]$, we have $\frac{\frac{\pi}{5\sin(\pi/5)}}{\sin\gamma\tan\frac{\beta}{2}+\cos\gamma} \leq \frac{\pi}{5\sin(\pi/5)\cos(\pi/5)} \leq \rho$, as required. □