Strictly Localized Construction of Planar Bounded-Degree Spanners of Unit Disk Graphs

Iyad A. Kanj* Ljubomir Perković† Ge Xia‡

Abstract

We describe a new distributed and strictly-localized approach for constructing bounded degree planar spanners of unit disk graphs. Our results improve the best results in the literature in terms of the stretch factor and the degree bound, for both geometric and power spanners.

1 Introduction

Given a set of points embedded on the Euclidean plane, a unit disk graph U is defined as the graph consisting of all edges AB connecting points A and B whose Euclidean distance $|AB|$ is at most 1. The edge AB connecting points A and B is assumed to be embedded in the plane as the straight line segment AB. We define, for some fixed positive constant p, a cost function mapping each edge AB of U to the scalar $|AB|^p$, The cost of a simple path $A = M_0, M_1, ..., M_r = B$ in U is then:

$$\sum_{j=0}^{r-1} |M_j M_{j+1}|^p.$$

Among all paths between A and B, a path in U with the smallest cost is defined to be a *smallest cost path* and we denote its cost as $c_U^p(A, B)$. We define a subgraph H of U to be a *spanner* with respect to cost function c^p if there is a constant ρ such that for every two points $A, B \in U$ we have: $c_H^p(A, B) \le \rho c_U^p(A, B)$. The constant ρ is called the *stretch factor* of H. The two edge-cost functions most often used in the topology control literature are the geometric cost function with $p = 1$, in which case the spanner is called a *geometric spanner*, and the power cost function with $p \in [2, 5]$, in which case the spanner is called a *power spanner*. We focus here only on geometric and power spanners.

We consider the following fundamental network topology problem. Given a unit disk graph U modeling a network, construct a bounded-degree, planar spanner of U with small stretch factor. We require that the spanner construction be distributed and strictly localized in the following sense: the computation at each point depends solely on the computation of its 1-hop or 2-hop neighbors, and is independent of the computation at other points. This strict requirement makes the algorithm very applicable in ad-hoc sensor networks, as well as very robust: if a point is deleted, for example,

*School of CTI, DePaul University, 243 S. Wabash Avenue, Chicago, IL 60604-2301. Email: ikanj@cs.depaul.edu.

†School of CTI, DePaul University, 243 S. Wabash Avenue, Chicago, IL 60604-2301. Email: lperkovic@cs.depaul.edu.

‡Department of Computer Science, Lafayette College, Easton, PA 18042. Email: gexia@cs.lafayette.edu.

only the points within the vicinity of the deleted point will need to recompute their spanner links. We will show that even with the strict requirement, we are able to obtain near-optimal spanners.

The problem we are considering is motivated by the issue of topology control of ad-hoc and sensor networks [8, 9, 10]. Topology control is used, for example, to construct a topology of the original network that is amenable to routing or other networking applications. Some desirable properties of the resulting topology include (1) planarity: the underlying graph should be planar to allow, for example, guaranteed and efficient routing such as geometric routing ([1, 5]); (2) small stretch factor: for any two devices in the network there should be a path connecting them in the topology whose length is close to the length of the shortest path connecting the pair in the original network; (3) bounded degree: each device maintains links to only a constant number of devices in its communication range, thus minimizing interference and saving energy; and (4) constructible locally: the construction of the network topology should be distributed, simple, and strictly-localized as explained above (see [2] for a formal definition).

Our approach to this problem improves all previous approaches in several ways (see Section 2). Our approach gives the best bounds on the stretch factor and the point-degree combination—in some cases these bounds are near-optimal. Our approach is also strictly localized, and hence very robust: the computation at a point depends only on the coordinates of the point and its 1-hop (for power spanners) or 2-hop neighbors (for geometric spanners), and no propagation of information is needed. We note here that even though the previous approaches were localized in the sense that each point only communicates with its neighbors, a close examination shows that some of these algorithms process the points in some implicit order and require information propagation. In particular, the best previous approaches (in terms of stretch factor and degree bound) for computing power spanners relied on an acyclic orientation of the planar graph (see [4, 11]), which cannot be done locally without propagation of information.

To be able to achieve these bounds, we introduce a set of new geometric notions and tools that are of independent interest.

2 Comparison between our results and the previous best results

In terms of the computation of geometric spanners, Li et al. [6, 7] gave a distributed algorithm that constructs a planar geometric spanner of a unit disk graph with stretch factor $C_{del} \approx 2.42$; however, the spanner constructed can have unbounded degree. Wang and Li [12, 13] then showed how to construct a bounded-degree planar spanner of a unit disk graph with stretch factor $max\{\pi/2, 1 + \pi \sin(\alpha/2)\} \cdot C_{del}$ and degree bounded by $19 + 2\pi/\alpha$, where $0 < \alpha < 2\pi/3$ is a parameter. Very recently, Bose et. al [1] showed how to construct a planar geometric spanner of a unit disk graph with stretch factor $max\{\pi/2, 1 + \pi \sin(\alpha/2)\} \cdot C_{del}$ and degree bounded by $14 + 2\pi/\alpha$, for any $0 < \alpha \leq \pi/3$. This was the best bound on the stretch factor and the degree.

In our work we present a very simple and strictly-localized distributed algorithm that constructs a planar geometric spanner of the unit disk graph with stretch factor $(1 + 2\pi(k \cos(\pi/k))^{-1}) \cdot C_{del}$ and degree bounded by k, for any integer parameter $k \geq 14$. These results, in turn, significantly improve all the previous results of [1, 12, 13] on this problem in terms of: the upper bound on the stretch factor, the upper bound on the degree, and the simplicity and locality of the algorithm. In terms of efficiency, the presented algorithm exchanges no more than $O(n)$ messages in total, and

Table 1: Degree bounds (Δ) and stretch factors (ρ) of the algorithms with power exponent $p = 2$.

$k =$	10		11		12		13		14	
	Δ	ρ	Δ	ρ	Δ	ρ	Δ	ρ	Δ	ρ
Lower Bounds	10	1.32	11	1.27	12	1.23	13	1.20	14	1.18
`OurALGO`	10	1.39	11	1.32	12	1.27	13	1.23	14	1.20
SWLF-I	15	1.62	16	1.47	17	1.37	18	1.30	19	1.38
SWLF-II	10	8.48	11	5.48	12	4.31	13	3.70	14	3.32

runs in $O(\Delta \lg \Delta)$ local time at a point of degree Δ.

To compare our bounds to the previous best bounds in the literature, for a degree bound $k = 14$, our results imply a bound of at most 3.54 on the stretch factor, and as k approaches ∞, our bound on the stretch factor approaches C_{del}. The smallest degree bound derived by Bose et al. [1] is 20, and that corresponds to a stretch factor of at least 6.19. If Bose et al. [1] allow the degree bound to be arbitrarily large, then their bound on the stretch factor approaches $(\pi/2) \cdot C_{del} > 3.75$. On the other hand, the smallest degree bound derived in Wang et al. [12, 13] is 25, and that corresponds to a bound of 6.19 on the stretch factor. If Wang et al. [12, 13] allow the degree bound to be arbitrarily large, then their bound on the stretch factor approaches $(\pi/2) \cdot C_{del} > 3.75$. Therefore, even the worst bound of at most 3.54 on the stretch factor corresponding to our lowest bound on the degree $k = 14$, beats the best bound on the stretch factor of at least 3.75 corresponding to arbitrarily large degree in both Bose et al. [1] and Wang et al. [12, 13]!

For the problem of constructing power spanners of unit disk graphs, Wattenhofer et al. [14] derived algorithms with arbitrarily small stretch factor but unbounded degree. To bound the degree, their stretch factor needs to be at least 2. The best previous bounds (with respect to minimizing both the stretch factor and degree) are by Song, Wang, Li, and Frieder who proposed distributed localized algorithms [11] with guarantees on the maximum degree and stretch factors. Given a parameter $k > 6$, their first algorithm (referred to as SWLF-I) computes a power spanner of maximum degree $k + 5$ and maximum stretch factor $\rho = 1/(1 - (2 \sin \frac{\pi}{k})^p)$. In their second algorithm (referred to as SWLF-II), given a parameter $k > 8$, they obtain a bound of k for the maximum degree and $\rho = (\sqrt{2})^p/(1 - (2\sqrt{2} \sin \frac{\pi}{k})^p)$ for the stretch factor. In our work we show how to construct a planar power spanner of maximum degree k and with stretch factor $\rho = 1 + (2 \sin \frac{\pi}{k})^p$. *For the same degree bound, this bound on the stretch factor improves the previous best bounds by Song et al. [11] significantly*; see Table 1 for a comparison between these bounds. Furthermore, *we show that the upper bound on the stretch factor proved is tight* by proving that constructing a power spanner of a unit disk graph with degree bound of k and a stretch factor smaller than $\rho = 1 + (2 \sin \frac{\pi}{k+1})^p$ is not possible.

3 Our techniques and contributions

The basic idea is to start by constructing a sparse subgraph G of the unit disk graph U, not necessarily of bounded degree, and then to bound the degree of each point in this subgraph, without hurting the stretch factor by much. We give below an intuitive summary of the main techniques and contributions of our results. We omit the technical details and intermediate results for lack of space.

3.1 The underlying sparse subgraph of U

For the case of geometric spanners, the underlying subgraph is a version of the unit Delaunay graph introduced by [12, 13], called the *2-localized Delaunay* subgraph or shortly $LDel^{(2)}$. We let $G = LDel^{(2)}(U)$ in this case, and note that G has stretch factor $C_{del} \approx 2.42$ [12, 13].

For the case of power spanners, it turns out that the Gabriel graphs [3] used in the literature to obtain power spanners [11] do not lead to tight results. We introduce the notion of *generalized Gabriel graphs*, prove some elegant non-trivial structural results about them, and then use them as the underlying sparse subgraphs. Let A and B be two points in U. Define the region $D^p(A, B)$ to be the set of all points M in the plane satisfying $|MA|^p + |MB|^p \leq |AB|^p$. Note that $D^2(A, B)$ is the closed disk of diameter AB.

We prove that $D^p(A, B)$ is a closed convex set and that every point $M \in D^p(A, B) \setminus \{A, B\}$ satisfies $\angle AMB \geq \arccos(1 - 2^{-3/5})$.

Definition 3.1 *The* generalized Gabriel graph *of U with parameter $p \in [2, 5]$, denoted G^p, is defined to be the subgraph of U having the same point set as U, and such that an edge $AB \in U$ is also an edge in G^p if and only if there does not exist a point $M \in D^p(A, B) \setminus \{A, B\}$.*

Theorem 3.2 *Let U be a unit disk graph, $p \in [2, 5]$ a constant, and G^p the generalized Gabriel graph of U with parameter p. Then G^p is a connected planar subgraph of U with stretch factor of 1.*

In the case of power spanners, we let $G = G^p$, and we note that by the above theorem G has stretch factor 1.

Furthermore, both $LDel^{(2)}(U)$ and the generalized Gabriel graphs can be easily constructed using a strictly-localized distributed algorithm.

3.2 Canonical paths

Let G be the sparse subgraph of U constructed above. Let BC and CA be edges of G such that $\angle BCA \leq 2\pi/k$, where k is the parameter used to bound the degree. Suppose further that BC is the shortest edge in G within the angular sector $\angle BCA$. In our work we show that either edge $AB \in G$, or there exists a *canonical path* $\mathcal{P}$ in G from A to B of small cost. This path is constructed recursively. For the case of geometric spanners, this path is the concatenation of several paths, called the *outward* paths, between points on the convex hull of the set of points residing in $\triangle ABC$.

For the case of geometric spanners, we bound the length of $\mathcal{P}$ by $2\pi(k \cos \frac{\pi}{k})^{-1}|CB|$. For the case of power spanners, we bound the length of $\mathcal{P}$ by $(2 \sin \frac{\pi}{k})^p |CB|$. We prove that the region determined by edges BC, CA and the path $\mathcal{P}$ is very simple: It is closed region of the plane in which the only possible edges join C to vertices on $\mathcal{P}$. More importantly, we prove that the internal angles on the path $\mathcal{P}$ are large, which is an essential property for our results to carry through.

3.3 The modified Yao construction

To construct a bounded-degree subgraph G' of G with small stretch factor, we try to select the edges on the canonical paths and include them in G'. Our results show that it is always possible to select at most k edges out of every point such that the edges on the canonical paths are selected as well. The crucial argument to why this works is that the angles on the canonical paths are large.

Almost all the approaches to bound the spanner degree are based on constructing a Yao subgraph of some planar subgraph of U. Given integer parameter $k > 6$, the *Yao subgraph* [15] of a directed graph G (embedded in the plane) is constructed as follows. At every point M in G, place k equally-separated rays out of M (arbitrarily defined), thus creating k closed cones of size $2\pi/k$ each. Then, the shortest edge in G out of M (if any) in each cone is added to the Yao subgraph of G.

It is well-known that the Yao construction as is does not yield a bounded degree subgraph. More importantly, the Yao construction does not always guarantee that edges forming large angles on the canonical paths are always selected. We give below an intuitive description of how we modify the Yao construction to achieve these goals.

First, every point selects the edges incident on it that form large angles. Then, the standard Yao construction is applied to the set of remaining incident edges. Finally, edges selected by both endpoints are kept in G'.

For geometric spanners, we can show that every point selects at most k edges, and that for every edge CB that was not selected in G' there is a selected edge $CA \in G'$ with $\angle BCA \leq 2\pi/k$, and such that the path CA followed by the canonical path from A to B has length at most $(1 + 2\pi(k \cos \frac{\pi}{k})^{-1})|CB|$, giving us a stretch factor of $(1 + 2\pi(k \cos \frac{\pi}{k})^{-1}) \cdot C_{del}$ for $k \geq 14$.

For power spanners, we modify the construction a little bit by first selecting incident edges that make large angles, then removing every edge with a certain small "buffer angle" from the selected edges, and finally applying the standard Yao construction to the remaining set of edges. With this modification, and using the notion of buffer angles, we can show that every point selects at most k edges, and that for every edge CB that was not selected in G' there is a selected edge $CA \in G'$ with $\angle BCA \leq 2\pi/k$, and such that the path CA followed by the canonical path from A to B has cost at most $(1 + (2 \sin \frac{\pi}{k})^p)|CB|$, giving a stretch factor of $1 + (2 \sin \frac{\pi}{k})^p$ for $k \geq 10$. We show that this ratio is near optimal by showing that the slightly smaller ratio of $1 + (2 \sin \frac{\pi}{(k+1)})^p$ is unattainable.

3.4 Simplicity, locality, and robustness

Besides improving all the previous results in terms of the stretch factor and degree bound, our algorithms are very simple and strictly localized: each point decides on the set of edges to keep based on the coordinates of its 1-hop neighbors (power spanner) and 2-hop neighbors (geometric spanner), and edges kept by both endpoints are selected in the spanner. The best previous algorithms (in terms of the stretch factor and degree bound) in the literature for power spanners required an acyclic orientation of the planar graph G before the edges in G' could be selected [11, 4]. Clearly, such an orientation cannot be done is such a strictly localized way.

This simplicity and strictly-localized nature of our algorithms make them very suitable for mobile networks, and very robust against topological changes, which we prove formally.

References

[1] P. Bose, M. Smid, and D. Xu. Diamond triangulations contain spanners of bounded degree. In *Proceedings of the 17th International Symposium on Algorithms and Computation*, volume 4288 of *Lecture Notes in Computer Science*, pages 173–182. Springer, 2006.

[2] Haowen Chan and Adrian Perrig. ACE: An emergent algorithm for highly uniform cluster formation. In *European Workshop on Wireless Sensor Networks (EWSN 2004)*, January 2004.

[3] K. Ruben Gabriel and Robert R. Sokal. A new statistical approach to geographic variation analysis. *Systematic Zoology*, 18(3):259–278, 1969.

[4] I. Kanj and L. Perkovic. Improved stretch factor for bounded-degree planar power spanners of wireless ad-hoc networks. In *Proceedings of the 2nd International Workshop on Algorithmic Aspects of Wireless Sensor Networks ALGOSENSOR*, volume 4240 of *Lecture Notes in Computer Science*, pages 95–106. Springer, 2006.

[5] E. Kranakis, H. Singh, and J. Urrutia. Compass routing on geometric networks. In *Proceeding of the 11th Canadian Conference on Computational Geometry*, volume 11, pages 51–54, 2005.

[6] X.-Y. Li, G. Calinescu, and P.-J. Wan. Distributed construction of planar spanner and routing for ad hoc wireless networks. In *Proceedings of the IEEE INFOCOM*, 2002.

[7] X.-Y. Li, G. Calinescu, P.-J. Wan, and Y. Wang. Localized delaunay triangulation with application in Ad Hoc wireless networks. *IEEE Transactions on Parallel and Distributed Systems.*, 14(10):1035–1047, 2003.

[8] R. Rajaraman. Topology control and routing in ad hoc networks: a survey. *SIGACT News*, 33(2):60–73, 2002.

[9] P. Santi. *Topology Control in Wireless Ad Hoc and Sensor Networks.* John Wiley & Sons, Ltd, 2005.

[10] C. Scheideler. Overlay networks for wireless ad hoc networks, 2005. http://www14.in.tum.de/personen/scheideler/papers/Overlay-IMA05.pdf.

[11] Wen-Zhan Song, Xiang-Yang Li Yu Wang, and Ophir Frieder. Localized algorithms for energy efficient topology in wireless ad hoc networks. *Mobile Networks and Applications*, 10(6):911–923, 2005.

[12] Y. Wang and X.-Y. Li. Localized construction of bounded degree and planar spanner for wireless ad hoc networks. In *Proceedings of the DIALM-POMC Joint Workshop on Foundations of Mobile Computing*, pages 59–68. ACM, 2003.

[13] Y. Wang and X.-Y. Li. Localized construction of bounded degree and planar spanner for wireless ad hoc networks. *MONET*, 11(2):161–175, 2006.

[14] R. Wattenhofer, E. L. Li, P. Bahl, and Y.-M. Wang. Distributed topology control for wireless multihop ad-hoc networks. In *INFOCOM*, pages 1388–1397, 2001.

[15] Andrew Chi-Chih Yao. On constructing minimum spanning trees in k-dimensional spaces and related problems. *SIAM Journal on Computing*, 11(4):721–736, 1982.