An Optimal Localized Approximation Scheme for Euclidean MST

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Abstract

We consider the problem of locally constructing a spanning subgraph that approximates the Euclidean minimum spanning tree of a unit disk graph. We show that for any $k \geq 2$ there exists a k-localized distributed algorithm that, given a unit disk graph U in the plane, constructs a planar subgraph of U containing a Euclidean MST on $V(U)$, whose degree is at most 6, and whose total weight is at most $1 + \frac{2}{k-1}$ times the weight of the Euclidean MST on $V(U)$. We prove that this approximation bound is tight by showing that, for any $\epsilon > 0$, no k-localized algorithm can construct a spanning subgraph of U whose total weight is less than $1 + \frac{2}{k-1} - \epsilon$ times the weight of a Euclidean MST on $V(U)$. En route to our results, we prove a nice result about weighted planar graphs.

1 Introduction

The minimum spanning tree (MST) problem is one of the most fundamental computational problems that has motivated a tremendous amount of research in the last 70 years. The Euclidean MST (EMST) problem is an important and natural restriction of the general MST problem due to its applications in computational geometry, graphics, and networking. In this restriction, the graph vertices are points in the d-dimensional Euclidean space and the *weight* of an edge is the Euclidean distance between its endpoints.

Most of the previous algorithms for computing EMST are either centralized or, if distributed, use propagation of information which makes them unsuitable for many applications in emerging distributed systems technologies such as wireless ad-hoc and sensor networks. For these applications, an alternative distributed model of computation has been defined. In a wireless ad-hoc environment the network is commonly modeled as a *unit disk graph* (UDG) in the Euclidean plane. The points of the UDG correspond to the mobile wireless devices, and its edges connect pairs of points whose corresponding devices are in each other's transmission range equal to one unit. A distributed algorithm in this type of network is said to be k-*localized* [5] if the computation at each point depends only on the information available at points that are at most k hops (edges) away.

Unfortunately, it is impossible to construct an EMST, or even a spanning tree, of a UDG using a k-localized distributed algorithm (for any $k < n/2$) because long cycles cannot be detected locally. We will say that a spanning subgraph of U has *low* weight if its weight is at most $c \cdot wt(\text{EMST})$ for some small constant $c \geq 1$. We are therefore addressing the following problem:

> Design a k-localized distributed algorithm that, given a UDG U embedded in the plane, constructs a low weight bounded-degree spanning subgraph G of U.

This problem has been considered in several recent papers. Li, Houa and Sha [2] were the first to propose a localized algorithm that constructs a spanning subgraph G whose degree is bounded by 6; however, it was not proved that G has low weight. Li [3] showed that it is necessary for every point to use its 2-neighborhood information in the construction of any low weight spanning subgraph. Li [3] and Li, Wang, and Song [4] then generalized the approach in [2] by having each point construct an EMST of the subgraph induced by its k-neighborhood, for $k \geq 2$. All these trees are then combined to form the desired subgraph of U, referred to as $RRNG_k(U)$ in this paper. They

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proved that $RRNG_k(U)$ can be constructed by a k-localized algorithm and contains an EMST. Furthermore, they showed that $RRNG_k(U)$ is planar, has degree bounded by 6, and has weight bounded by $c \cdot wt(\text{EMST})$; however, the undetermined constant c in the above bound was imported from a more general result by Das, Narasimhan, and Salowe [1], and can be quite large.

We outline our results in the following section.

2 Our results

We start by showing a more general result about weighted planar graphs that may be of independent interest. Note that the edge weights in this theorem can be arbitrarily positive numbers (i.e., not necessarily the Euclidean weight). The result can also be extended to graphs of bounded genus.

Theorem 2.1 *Let G be a weighted connected planar graph with nonnegative weights satisfying the following property: for every cycle C in G and every edge $e \in C$, $wt(C) \geq \lambda \cdot wt(e)$ for some constant $\lambda > 2$. Then $wt(G) \leq (1 + \frac{2}{\lambda - 2}) \cdot wt(T)$, where T is a MST of G.*

Proof: [sketch] Embed G in the plane and consider a MST T of G. Let R be the set of edges in $E(G) - E(T)$. Call an edge $e \in E(T)$ a *tree edge* and an edge $e \in R$ a *non-tree edge*. We bound the total weight of the non-tree edges in terms of the total weight of the tree edges. Observe that every non-tree edge e induces a unique cycle in the graph $T + e$ called the *fundamental cycle* of e. A partial order can be defined on the set of edges of G based on these fundamental cycles. Using this partial order, we design a "charging scheme" that charges the weight of every edge in R to the edges of T and hence bound the the total weight of the edges in $E(G)$ in terms of $wt(T)$. $\square$

The above bound is tight as shown in the following theorem:

Theorem 2.2 *For any integer $\lambda > 2$ and any $\epsilon > 0$, there exists a weighted planar graph G satisfying $wt(C) \geq \lambda \cdot wt(e)$ for any cycle C and any edge $e \in C$, such that $wt(G) \geq (1 + \frac{2}{\lambda - 2} - \epsilon) \cdot wt(T)$, where T is a MST of G.*

Let U be a unit disk graph in the plane. We apply the above results to $RRNG_k(U)$.

Lemma 2.3 *For any cycle C in $RRNG_k(U)$ and any edge $e \in C$, $wt(C) \geq (k + 1) \cdot wt(e)$.*

Corollary 2.4 *For any integer $k \geq 2$, we have $wt(RRNG_k(U)) \leq (1 + 2/(k - 1)) \cdot wt(T)$, where T is an EMST of $V(U)$.*

Proof: Follows from Theorem 2.1, Lemma 2.3, and the planarity of $RRNG_k(U)$. $\square$

Finally, we summarize our main results:

Theorem 2.5 *For any integer $k \geq 2$, there is a k-localized algorithm that, given a unit disk graph U in the plane, constructs a planar spanning subgraph of U containing an EMST of $V(U)$, whose degree is bounded by 6, and whose total weight is bounded by $(1 + 2/(k - 1)) \cdot wt(EMST)$. Moreover, for any $k \geq 2$ and any $\epsilon > 0$, there exists a unit disk graph U such that no k-localized algorithm can construct a spanning subgraph of U whose weight is less than $(1 + \frac{2}{k-1} - \epsilon) \cdot wt(EMST)$.*

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